CSP Exercise 02 Solution

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Exercise 2.1

(a) The strongly connected components are the subgraphs induced by following sets of vertices.

- $V_1 = \{v_1, v_2, v_5, v_6\}$
- $V_2 = \{v_3, v_4\}$

(b) The cliques are the the complete subgraphs induced by the following sets of vertices.

- $V_1 = \{v_1, v_2, v_5\}$
- $V_{2-8} \in E$
- $V_{9-14} \in \{\{v_i\} \mid v_i \in V\}$

Exercise 2.2

We define the following undirected graph $G = \langle W, E\rangle$ with $\{w_i, w_j\} \in E$ iff w_i and w_j dislike each other. Additionally we define two sets $B_1 = B_2 = \{\}$ for the buildings and an auxiliary set $F = \{\}$ for remembering *expanded* nodes.

1. For all $w_i \in W$ with $w_i \notin \bigcup E$ add w_i to B_1 and F, i.e. $B_1 = B_1 \cup \{w_i\}$ and $F = F \cup \{w_i\}$. This way we assign all workers, who are liked by everyone to building B_1.
2. If $F = W$ terminate, we have found a solution.
   Otherwise select any $w_i \notin F$ and add it to B_1 and F, i.e. $B_1 = B_1 \cup \{w_i\}$ and $F = F \cup \{w_i\}$.
   Add all its disliked neighbours to B_2, i.e. $B_2 = B_2 \cup \{w_j \mid \{w_i, w_j\} \in E\}$.
   If $B_1 \cap B_2 \neq \emptyset$ terminate, there was a conflict and therefore no possible solution.
3. If there a $w_i \notin F$ with $w_i \in B_1 \cup B_2$ select it, otherwise continue with 2.
   Add all its disliked neighbours to the other building, i.e. $w_i \in B_1 \implies B_2 = B_2 \cup \{w_j \mid (w_i, w_j) \in E\}$, $w_i \in B_2 \implies B_1 = B_1 \cup \{w_j \mid (w_i, w_j) \in E\}$ and $F = F \cup \{w_i\}$.
   If $B_1 \cap B_2 \neq \emptyset$ terminate, there was a conflict and therefore no possible solution.
   Otherwise continue with 3.

This is essentially a random walk through the graph with some book keeping. Assumming that set containment can be tested in $\mathcal{O}(1)$ and set intersection in $\mathcal{O}(n)$ we have an asymptotic complexity of $\mathcal{O}(|W| \cdot |E|)$ (this includes the optimisation, that the intersection test is only done once when $F = W$ holds).

Exercise 2.3

(a) $VertexCover \in NP$ because by guessing the vertex set V', we can iterate over all $\{v_1, v_2\} \in E$ and check if $v_1 \in V'$ or $v_2 \in V'$ holds in polynomial time.
We show $Clique \leq_P VertexCover$. Given an undirected graph $G = \langle V, E\rangle$ and $k \in N$ we build a graph $G_2 = \langle V_2, E_2\rangle$ so that there is a clique $V' \subseteq V$ with $|V'| \geq k$ iff there is a vertex cover $V_2' \subseteq V_2$ with $|V_2'| \leq k_2$ given following construction:

- $V_2 = V$
- $E_2 = \{\{v_i, v_j\} \mid \{v_i, v_j\} \notin E\}$
- $k_2 = |V| - k$

If there is a vertex cover V_2' of size $\leq k_2$, i.e. $\forall e \in E : \exists v \in V_2' : v \in e$ then it follows that there must be a clique of at least size $|V| - k_2$ in the complementary graph and vice versa, because for each disconnected vertex pair in one graph, there is a connected pair in the other.
The construction takes linear time, hence it follows that $VertexCover \in NP\text{-}complete$. $\square$

(b) $DominatingSet \in NP$ because by guessing the vertex set V', we can iterate over all $v \in V$ and check if $v \in V'$ or $\exists v' \in V : \{v, v'\} \in E$ holds in polynomial time.
We show $VertexCover \leq_P DominatingSet$. Given an undirected graph $G = \langle V, E \rangle$ and $k \in N$ we build a graph $G_2 = \langle V_2, E_2 \rangle$ so that there is a vertex cover $V' \subseteq V$ with $|V'| \leq k$ iff there is a dominating set $V_2' \subseteq V_2$ with $|V_2| \leq k$ given following construction:

- $V_2 = V \cup \{v_e \mid e \in E\}$
- $E_2 = E \cup \{\{v, v_e\} \mid v, v_e \in V_2$ and $e \in E\}$

By introducing the additional nodes for all vertices in G, we make sure that the dominating set V_2', by covering all vertices in G_2, implicitely covers all edges in G. The other direction follows analogously, if a vertex cover V' covers all edges in G it follows that the same set covers all vertices in G_2.
The construction takes linear time, hence it follows that $DominatingSet \in NP\text{-}complete$. $\square$