CSP Exercise 03 Solution

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Exercise 3.1

The minimal network is $N_0 = \langle (v_1, v_2, v_3), (\{2,3\}, \{1,2,3,4\}, \{1,3,4\}), \{<_{(v_1,v_2)}, <_{(v_1,v_3)}, =_{(v_2,v_3)}\}\rangle$.

Exercise 3.2

(a) The network for the puzzle is *(we do not name the variables consecutively to keep a clear connection to the original puzzle fields)*

$N = \langle (v_1, v_2, v_3, v_8), (\{ATOLL\}, \{TOLL, HEAT\}, \{EAT\}, \{TOLL, HEAT\}), \{R_{v_1,v_2,v_8}, R_{v_2,v_3}, R_{v_3,v_8}\}\rangle$

with

$$R_{v_1,v_2,v_8} = \{(ATOLL, HEAT, TOLL)\}$$
$$R_{v_2,v_3} = \{(HEAT, EAT)\}$$
$$R_{v_3,v_8} = \{(EAT, TOLL)\}$$

(b) See figure 1.

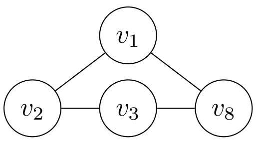

Figure 1: (3.2b) primal constraint graph of N

(c) See figure 2.

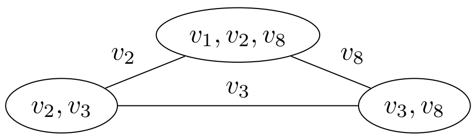

Figure 2: (3.2c) dual constraint graph of N

Exercise 3.3

There are more trivial examples with the required properties and it would be easier to prove the properties on them in a formal way. However, we tried to find an example, which works for all three parts of this exercise with minimal modifications. We traded off formal proofs for intuitive understanding of the underlying differences between these relations.

(a) Let R_S be the relation with $S = (x_1, ..., x_n)$ for the constraint $x_1 > \sum_{i=2}^{n} x_i$ over the domains $D_i = \{0, 1, 2\}$, i.e., $R_S = \{(1, 0, ..., 0)\} \cup \{(2, (x_i)_{1<i\leq n}) \mid \sum_{i=2}^{n} x_i < 2\}$ or more graphically $R_S = \{(1, 0, ..., 0), (2, 0, ..., 0, 1, 0, ..., 0), (2, 1, 0, ..., 0), (2, 0, ..., 0, 1)\}$.

The projection network of R_S is by definition $Proj(R_S) = \langle S, D_i = \pi_{x_i}(R_S), R'_{x_i,x_j} = \pi_{x_i,x_j}(R_S)\rangle$ with $R'_{x_1,x_j} = \{(1, 0), (2, 0), (2, 1)\}$ and for $i \neq 1$ holds $R'_{x_i,x_j} = \{(0, 0), (0, 1), (1, 0)\}$.

The limited domains impose transitive relations $x_1 > x_i$ for all $1 < i \leq n$ and $x_i = x_j = 0 \vee x_i \neq x_j$ for $i \neq 1$. Since this imposed relations essentially reduce the contraint to the mentioned binary relations R' it follows that all solutions of the projection network are also in the relation, i.e., $Sol(Proj(R_S)) = R_S$.

All projections of R_S to a subset of S, so that x_1 is contained, basically reduces the network to the projection size, all properties remain unchanged, therefore the projection network of the projection of R_S has only solutions which are in the projected relation (and vice versa).

All projections of R_S to a subset of S without x_1 mean a reduction to the constraint $2 > \sum_{i=1}^{m} x_i$ where m is the projection size. This has obviously also a binary representation with $R'_{x_i,x_j} = \{(0, 0), (1, 0), (0, 1)\}$. Again it follows that the projection network of this projection has only solutions which are in the projected relation (and vice versa).

Given that the relation R_S itself and all its possible projections have binary representations, it holds that R_S is binary decomposable. □

(c) We will do part (c) before (b) because here we use the same constraint as in part (a). However, we will expand the domains with $D_i = \{0, 1, 2, 3\}$. This gives us the relation $R_S = \{(x_1, (x_i)_{1<i\leq n}) \mid \sum_{i=2}^{n} x_i < x_1\}$.

This relation has no binary representation, because no set of binary relations can enforce the constraint for $x_1 = 3$. E.g. $(3, 1, 1, 1, 0, ..., 0)$ is a solution of $Proj(R_S)$, because it is consistent with the constraints $R'_{x_1,x_j} = \{(1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2)\}$ and for $i \neq 1$ $R'_{x_i,x_j} = \{(0, 0), (1, 0), (0, 1), (1, 1), (2, 0)\}$ while it is not contained in R_S. It follows that $R_S \subseteq Sol(Proj(R_S))$ and $R_S \neq Sol(Proj(R_S))$.

We could have shown the same effect by leaving the domains untouched, but changing the constraint to $x_1 \geq \sum_{i=2}^{n} x_i$ instead, which has the same effect, it requires a systematic (here ternary) view over all variables. □

(b) This time we modify the relation like this

$$R_S = \{(x_1, (x_i)_{1<i\leq \frac{n-1}{2}}, (x_j)_{\frac{n-1}{2}<j\leq n}) \mid \sum_{i=2}^{\frac{n-1}{2}} x_i < x_1 < \sum_{j=\frac{n-1}{2}+1}^{n} x_j\}$$

with domains $D_i = \{0, 1, 2\}$ again.

This modification does not change the property of the relation to have a binary representation, because solutions to the binary constraints are exactly the variable assignments from the relation (again due to the transitive relations over x_1).

However, if we use a projection without x_1, we lose our pivot variable for the otherwise separted sets of variables. Solutions to such a projection network induced by the projected relation without x_1 will again lack the systematic view over the separated sets of variables and their sums, which yields possible solutions which are inconsistent with the projected relation. From this follows that our modified R_S has a binary representation but is not binary decomposable. □